THE ELEMENTS

OF

COORDINATE GEOMETRY.



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BY

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CHESTNUT HILL, MASS.

MATH. DEPT.

London:
MACMILLAN AND CO.
AND NEW YORK.
1895

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Cambridge:
PRINTED BY J. & C. F. CLAY,
AT THE UNIVERSITY PRESS.

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PREFACE.

IN the following work I have tried to present the elements of Coordinate Geometry in a manner suitable for Beginners and Junior Students. The present book only deals with Cartesian and Polar Coordinates. Within these limits I venture to hope that the book is fairly complete, and that no propositions of very great importance have been omitted.

The Straight Line and Circle have been treated more fully than the other portions of the subject, since it is generally in the elementary conceptions that beginners find great difficulties.

There are a large number of Examples, over 1100 in all, and they are, in general, of an elementary character. The examples are especially numerous in the earlier parts of the book.

I am much indebted to several friends for reading portions of the proof sheets, but especially to Mr W. J. Dobbs, M.A. who has kindly read the whole of the book and made many valuable suggestions.

For any criticisms, suggestions, or corrections, I shall be grateful.

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July 4, 1895.

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ERRATA.

Page 87, Ex. 27, line 4. *For* "*R*" *read* "*S*."

,, 235, Ex. 18, line 3. *For* "odd" *read* "even."

,, ,, ,, ,, line 5. *Dele* "and Page 37, Ex. 15."

,, 282, Ex. 3. *For* "transverse" *read* "conjugate."

CHAPTER I.

INTRODUCTION.

SOME ALGEBRAIC RESULTS.

1. Quadratic Equations. The roots of the quadratic equation

$$ax^2 + bx + c = 0$$

may easily be shewn to be

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

They are therefore real and unequal, equal, or imaginary, according as the quantity $b^2 - 4ac$ is positive, zero, or negative,

i.e. according as $b^2 \gtreqless 4ac$.

2. *Relations between the roots of any algebraic equation and the coefficients of the terms of the equation.*

If any equation be written so that the coefficient of the highest term is unity, it is shewn in any treatise on Algebra that

(1) the sum of the roots is equal to the coefficient of the second term with its sign changed,

(2) the sum of the products of the roots, taken two at a time, is equal to the coefficient of the third term,

(3) the sum of their products, taken three at a time, is equal to the coefficient of the fourth term with its sign changed,

and so on.

Ex. 1. If α and β be the roots of the equation

$$ax^2+bx+c=0, \text{ i.e. } x^2+\frac{b}{a}x+\frac{c}{a}=0,$$

we have $$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}.$$

Ex. 2. If α, β, and γ be the roots of the cubic equation

$$ax^3+bx^2+cx+d=0,$$

i.e. of $$x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0,$$

we have $$\alpha+\beta+\gamma=-\frac{b}{a},$$

$$\beta\gamma+\gamma\alpha+\alpha\beta=\frac{c}{a},$$

and $$\alpha\beta\gamma=-\frac{d}{a}.$$

3. It can easily be shewn that the solution of the equations

$$a_1x+b_1y+c_1z=0,$$

and $$a_2x+b_2y+c_2z=0,$$

is $$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}.$$

Determinant Notation.

4. The quantity $\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix}$ is called a determinant of the second order and stands for the quantity $a_1b_2-a_2b_1$, so that

$$\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix}=a_1b_2-a_2b_1.$$

Exs. (i) $\begin{vmatrix} 2, & 3 \\ 4, & 5 \end{vmatrix}=2\times5-4\times3=10-12=-2;$

(ii) $\begin{vmatrix} -3, & -4 \\ -7, & -6 \end{vmatrix}=-3\times(-6)-(-7)\times(-4)=18-28=-10.$

5. The quantity $\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix}$ .....(1)

is called a determinant of the third order and stands for the quantity

$$a_1 \times \begin{vmatrix} b_2, & b_3 \\ c_2, & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1, & b_3 \\ c_1, & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1, & b_2 \\ c_1, & c_2 \end{vmatrix} \text{ ..... (2),}$$

i.e. by Art. 4, for the quantity

$$a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1),$$

i.e. $$a_1 (b_2c_3 - b_3c_2) + a_2 (b_3c_1 - b_1c_3) + a_3 (b_1c_2 - b_2c_1).$$

6. A determinant of the third order is therefore reduced to three determinants of the second order by the following rule:

Take in order the quantities which occur in the first row of the determinant; multiply each of these in turn by the determinant which is obtained by erasing the row and column to which it belongs; prefix the sign + and − alternately to the products thus obtained and add the results.

Thus, if in (1) we omit the row and column to which a_1 belongs, we have left the determinant $\begin{vmatrix} b_2, & b_3 \\ c_2, & c_3 \end{vmatrix}$ and this is the coefficient of a_1 in (2).

Similarly, if in (1) we omit the row and column to which a_2 belongs, we have left the determinant $\begin{vmatrix} b_1, & b_3 \\ c_1, & c_3 \end{vmatrix}$ and this with the − sign prefixed is the coefficient of a_2 in (2).

7. Ex. The determinant $\begin{vmatrix} 1, & -2, & -3 \\ -4, & 5, & -6 \\ -7, & 8, & -9 \end{vmatrix}$

$$= 1 \times \begin{vmatrix} 5, & -6 \\ 8, & -9 \end{vmatrix} - (-2) \times \begin{vmatrix} -4, & -6 \\ -7, & -9 \end{vmatrix} + (-3) \times \begin{vmatrix} -4, & 5 \\ -7, & 8 \end{vmatrix}$$

$$= \{5 \times (-9) - 8 \times (-6)\} + 2 \times \{(-4)(-9) - (-7)(-6)\} - 3 \times \{(-4) \times 8 - (-7) \times 5\}$$

$$= \{-45 + 48\} + 2\{36 - 42\} - 3\{-32 + 35\}$$

$$= 3 - 12 - 9 = -18.$$

8. The quantity $\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix}$

is called a determinant of the fourth order and stands for the quantity

$$a_1 \times \begin{vmatrix} b_2, & b_3, & b_4 \\ c_2, & c_3, & c_4 \\ d_2, & d_3, & d_4 \end{vmatrix} - a_2 \times \begin{vmatrix} b_1, & b_3, & b_4 \\ c_1, & c_3, & c_4 \\ d_1, & d_3, & d_4 \end{vmatrix}$$

$$+ a_3 \times \begin{vmatrix} b_1, & b_2, & b_4 \\ c_1, & c_2, & c_4 \\ d_1, & d_2, & d_4 \end{vmatrix} - a_4 \times \begin{vmatrix} b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \\ d_1, & d_2, & d_3 \end{vmatrix},$$

and its value may be obtained by finding the value of each of these four determinants by the rule of Art. 6.

The rule for finding the value of a determinant of the fourth order in terms of determinants of the third order is clearly the same as that for one of the third order given in Art. 6.

Similarly for determinants of higher orders.

9. A determinant of the second order has two terms. One of the third order has 3×2, *i.e.* 6, terms. One of the fourth order has $4 \times 3 \times 2$, *i.e.* 24, terms, and so on.

10. Exs. Prove that

(1) $\begin{vmatrix} 2, & -3 \\ 4, & 8 \end{vmatrix} = 28.$ (2) $\begin{vmatrix} -6, & 7 \\ -4, & -9 \end{vmatrix} = 82..$ (3) $\begin{vmatrix} 5, & -3, & 7 \\ -2, & 4, & -8 \\ 9, & 3, & -10 \end{vmatrix} = -98.$

(4) $\begin{vmatrix} 9, & 8, & 7 \\ 6, & 5, & 4 \\ 3, & 2, & 1 \end{vmatrix} = 0.$ (5) $\begin{vmatrix} -a, & b, & c \\ a, & -b, & c \\ a, & b, & -c \end{vmatrix} = 4abc.$

(6) $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$

Elimination.

11. Suppose we have the two equations

$$a_1x + a_2y = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$b_1x + b_2y = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

between the two unknown quantities x and y. There must be some relation holding between the four coefficients a_1, a_2, b_1, and b_2. For, from (1), we have

$$\frac{x}{y} = -\frac{a_2}{a_1},$$

and, from (2), we have $\dfrac{x}{y} = -\dfrac{b_2}{b_1}$.

Equating these two values of $\dfrac{x}{y}$ we have

$$\frac{b_2}{b_1} = \frac{a_2}{a_1},$$

i.e. $$a_1b_2 - a_2b_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The result (3) is the condition that both the equations (1) and (2) should be true for the same values of x and y. The process of finding this condition is called the eliminating of x and y from the equations (1) and (2), and the result (3) is often called the eliminant of (1) and (2).

Using the notation of Art. 4, the result (3) may be written in the form $\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix} = 0.$

This result is obtained from (1) and (2) by taking the coefficients of x and y in the order in which they occur in the equations, placing them in this order to form a determinant, and equating it to zero.

12. Suppose, again, that we have the three equations

$$a_1x + a_2y + a_3z = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$b_1x + b_2y + b_3z = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
and $$c_1x + c_2y + c_3z = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

between the three unknown quantities x, y, and z.

By dividing each equation by z we have three equations between the two unknown quantities $\frac{x}{z}$ and $\frac{y}{z}$. Two of these will be sufficient to determine these quantities. By substituting their values in the third equation we shall obtain a relation between the nine coefficients.

Or we may proceed thus. From the equations (2) and (3) we have

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{b_3c_1 - b_1c_3} = \frac{z}{b_1c_2 - b_2c_1}.$$

Substituting these values in (1), we have

$$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = 0 \ldots (4).$$

This is the result of eliminating x, y, and z from the equations (1), (2), and (3).

But, by Art. 5, equation (4) may be written in the form

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} = 0.$$

This eliminant may be written down as in the last article, *viz.* by taking the coefficients of x, y, and z in the order in which they occur in the equations (1), (2), and (3), placing them to form a determinant, and equating it to zero.

13. Ex. *What is the value of a so that the equations*

$$ax + 2y + 3z = 0, \quad 2x - 3y + 4z = 0,$$

and

$$5x + 7y - 8z = 0$$

may be simultaneously true?

Eliminating x, y, and z, we have

$$\begin{vmatrix} a, & 2, & 3 \\ 2, & -3, & 4 \\ 5, & 7, & -8 \end{vmatrix} = 0,$$

i.e. $a[(-3)(-8) - 4 \times 7] - 2[2 \times (-8) - 4 \times 5] + 3[2 \times 7 - 5 \times (-3)] = 0,$

i.e.

$$a[-4] - 2[-36] + 3[29] = 0,$$

so that

$$a = \frac{72 + 87}{4} = \frac{159}{4}.$$

14. If again we have the four equations

$$a_1x + a_2y + a_3z + a_4u = 0,$$

$$b_1x + b_2y + b_3z + b_4u = 0,$$

$$c_1x + c_2y + c_3z + c_4u = 0,$$

and $$d_1x + d_2y + d_3z + d_4u = 0,$$

it could be shewn that the result of eliminating the four quantities x, y, z, and u is the determinant

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix} = 0.$$

A similar theorem could be shewn to be true for n equations of the first degree, such as the above, between n unknown quantities.

It will be noted that the right-hand member of each of the above equations is zero.

CHAPTER II.

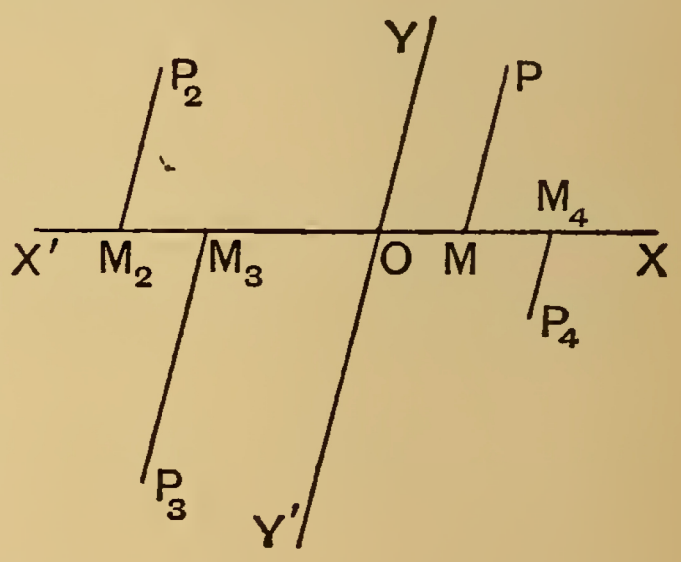
COORDINATES. LENGTHS OF STRAIGHT LINES AND AREAS OF TRIANGLES.

15. Coordinates. Let OX and OY be two fixed straight lines in the plane of the paper. The line OX is called the axis of x, the line OY the axis of y, whilst the two together are called the axes of coordinates.

The point O is called the origin of coordinates or, more shortly, the origin.

From any point P in the plane draw a straight line parallel to OY to meet OX in M.

The distance OM is called the Abscissa, and the distance MP the Ordinate of the point P, whilst the abscissa and the ordinate together are called its Coordinates.



Distances measured parallel to OX are called x, with or without a suffix, (*e.g.* x_1, x_2... x', x'',...), and distances measured parallel to OY are called y, with or without a suffix, (*e.g.* y_1, y_2,... y', y'',...).

If the distances OM and MP be respectively x and y, the coordinates of P are, for brevity, denoted by the symbol (x, y).

Conversely, when we are given that the coordinates of a point P are (x, y) we know its position. For from O we have only to measure a distance OM $(=x)$ along OX and

then from M measure a distance MP $(=y)$ parallel to OY and we arrive at the position of the point P. For example in the figure, if OM be equal to the unit of length and $MP=2OM$, then P is the point (1, 2).

16. Produce XO backwards to form the line OX' and YO backwards to become OY'. In Analytical Geometry we have the same rule as to signs that the student has already met with in Trigonometry.

Lines measured parallel to OX are positive whilst those measured parallel to OX' are negative; lines measured parallel to OY are positive and those parallel to OY' are negative.

If P_2 be in the quadrant YOX' and P_2M_2, drawn parallel to the axis of y, meet OX' in M_2, and if the numerical values of the quantities OM_2 and M_2P_2 be a and b, the coordinates of P are $(-a$ and $b)$ and the position of P_2 is given by the symbol $(-a, b)$.

Similarly, if P_3 be in the third quadrant $X'OY'$, both of its coordinates are negative, and, if the numerical lengths of OM_3 and M_3P_3 be c and d, then P_3 is denoted by the symbol $(-c, -d)$.

Finally, if P_4 lie in the fourth quadrant its abscissa is positive and its ordinate is negative.

17. Ex. *Lay down on paper the position of the points*

(i) $(2, -1)$, (ii) $(-3, 2)$, and (iii) $(-2, -3)$.

To get the first point we measure a distance 2 along OX and then a distance 1 parallel to OY'; we thus arrive at the required point.

To get the second point, we measure a distance 3 along OX', and then 2 parallel to OY.

To get the third point, we measure 2 along OX' and then 3 parallel to OY'.

These three points are respectively the points P_4, P_2, and P_3 in the figure of Art. 15.

18. When the axes of coordinates are as in the figure of Art. 15, not at right angles, they are said to be Oblique Axes, and the angle between their two positive directions OX and OY, *i.e.* the angle XOY, is generally denoted by the Greek letter ω.

In general, it is however found to be more convenient to take the axes OX and OY at right angles. They are then said to be Rectangular Axes.

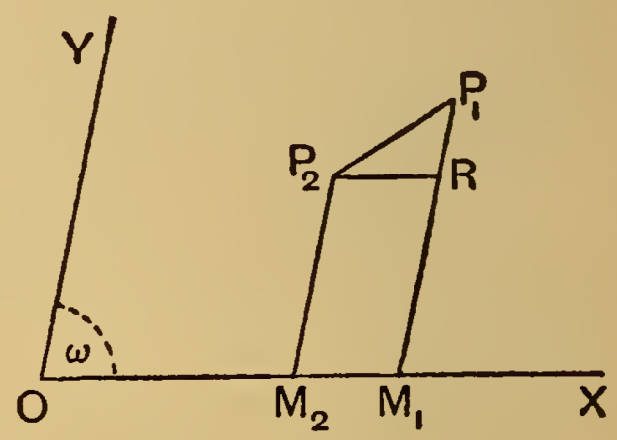
It may always be assumed throughout this book that the axes are rectangular unless it is otherwise stated.

19. The system of coordinates spoken of in the last few articles is known as the Cartesian System of Coordinates. It is so called because this system was first introduced by the philosopher Des Cartes. There are other systems of coordinates in use, but the Cartesian system is by far the most important.

20. *To find the distance between two points whose coordinates are given.*

Let P_1 and P_2 be the two given points, and let their coordinates be respectively (x_1, y_1) and (x_2, y_2).

Draw P_1M_1 and P_2M_2 parallel to OY, to meet OX in M_1 and M_2. Draw P_2R parallel to OX to meet M_1P_1 in R.



Then

$$P_2R = M_2M_1 = OM_1 - OM_2 = x_1 - x_2,$$

$$RP_1 = M_1P_1 - M_2P_2 = y_1 - y_2,$$

and $$\angle P_2RP_1 = \angle OM_1P_1 = 180° - P_1M_1X = 180° - \omega.$$

We therefore have [*Trigonometry*, Art. 164]

$$P_1P_2^2 = P_2R^2 + RP_1^2 - 2P_2R \,.\, RP_1 \cos P_2RP_1$$

$$= (x_1 - x_2)^2 + (y_1 - y_2)^2 - 2\,(x_1 - x_2)\,(y_1 - y_2) \cos (180° - \omega)$$

$$\mathbf{= (x_1 - x_2)^2 + (y_1 - y_2)^2 + 2\,(x_1 - x_2)\,(y_1 - y_2) \cos \omega} \ldots (1).$$

If the axes be, as is generally the case, at right angles, we have $\omega = 90°$ and hence $\cos \omega = 0$.

The formula (1) then becomes

$$P_1P_2^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$