Contents

Preface v

1 Introductory Problems 1
- 1.1 The Factorial 1
- 1.2 Fibonacci's Sequence: A Two-Term Recursion 9
- 1.3 The Tower of Hanoi 23
- 1.4 Derangements 27
- 1.5 The Bisection Method 32
- 1.6 Gambling Boldly 35
- 1.7 The Josephus Problem 37
- Exercises for Sections 1.1 to 1.7 42

2 Algorithms in Number Theory 47
- 2.1 Greatest Common Divisor 47
- Exercises 57
- 2.2 Representing n in the Form $x^2 + y^2$ 61
- 2.3 Pythagorean Triples 62
- 2.4 Counting the Lattice Points in a Ball 63
- Exercises for Sections 2.2 to 2.4 66
- 2.5 Sieves 67
- Exercises for Section 2.5.4 72
- Exercises for Section 2.5 73
- 2.6 Rotation of a List 77
- 2.7 Partitions 79
- 2.8 The Money Changing Problem 81
- 2.9 An Abstractly Defined Set 82
- 2.10 Recursive Pile Splitting 84
- 2.11 The Stern-Brocot Sequence 85
- 2.12 The Best Rational Approximation 90
- Exercises for Sections 2.6 to 2.12 91
- 2.13 Primes 95
- 2.14 n as a Sum of Four Squares 109

Exercises for Sections 2.13 to 2.14 111
2.15 The Maximum of a Unimodal Function 113
2.16 A Very Difficult Problem 117
Exercises 118

Additional Exercises for Chapters 1 to 2 120

3 Probability 127
3.1 A Pseudo-Random Number Generator 127
3.2 Python's PRNGs 129
3.3 Random Walks 130
3.4 Problems Involving Random 1D Walking 132
3.5 Self Avoiding Random Walks 140
Exercises 143
3.6 Finding Geometric Probabilities by Simulation 144
Exercises 148
3.7 The Bertrand Paradox 151
3.8 The Broken Stick Problem 155
3.9 The Monty Hall Problem 158
3.10 Random Choice of an s-Subset from an n-Set 160
Exercises 163
3.11 Coin Tosses in Software 163
Exercise 168
3.12 Shift Registers: More Coin Tosses 168
Exercises 170
3.13 Random Sequence Generation by Cellular Automata 172
Exercise 173
3.14 The Period Finding Problem 173
3.15 Shrinking Squares 175
Exercises 180

Additional Exercises for Chapters 1 to 3 182

4 Statistics 187
4.1 The Subset Sum Problem (Part 1) 187
4.2 Matched Pairs 190
4.3 P-value by Simulation 194
4.4 P-value by Computation 195
Exercises for Sections 4.1 to 4.4 196
4.5 The Two Sample Problem of Wilcoxon 197
4.6 The General Two Sample Test 204
Exercises for Sections 4.5 to 4.6 205
4.7 Kendall's Rank Correlation 206

Exercises 211
4.8 The Binomial Distribution 212
Exercises 216
4.9 Hypergeometric Distribution 218
Exercises 222
4.10 Fishing Laws 224
Exercises 225
Exercise 227
4.11 Probability of Not Being Lonesome 227
4.12 Runs 228
Exercises 235
4.13 Waiting for a Complete Set 236
Exercises 239
4.14 The Birthday Problem (and Factoring) 240
Exercise 246

Additional Exercises for Chapters 3 to 4 247

Additional Exercises for Chapters 1 to 4 250

5 Combinatorial Algorithms 251
5.1 Sorting 251
Exercises 260
5.2 The kth Smallest Element of a List 264
5.3 Binary Search 265
Exercises 270
5.4 Binary Questions 270
Exercises 270
5.5 Recreational Search 271
Exercises 286
5.6 Permutations 288
Exercises 292
5.7 Coupled Difference Equations 294
Exercises 296
5.8 Games 296
Exercises 302
5.9 The Subset Sum Problem (Part 2) 304
Exercises 314

6 Numerical Algorithms 317
6.1 Powers with Integer and Real Exponents 317
Exercises 319
6.2 Design of a Square Root Procedure 319

	Exercises	323
6.3	The Natural Logarithm	327
6.4	The Inverse Trigonometric Functions	329
	Exercise	331
6.5	The Function exp()	331
6.6	The Cosine	332
6.7	One Million Decimals of e and π	334
	Exercises	353
6.8	Least-Squares Curve Fitting	353
	Exercises	363
6.9	Polygon Circumscribing	363
6.10	Archimedes' Integration of the Parabola	365
6.11	Numerical Integration	367
6.12	Richardson Extrapolation	372
6.13	Romberg Integration	373
	Exercises	376
6.14	Problems of Pursuit	376
	Exercises	385
6.15	A Rumor Spreads Through the Town	387
	Exercises	391
6.16	Eat and be Eaten: the Struggle for Existence	392
	Exercises	395

Additional Exercises for Chapters 1 to 6 396

7 Fractals 403

	Why Study Fractals?	403
7.1	Introduction	404
7.2	The von Koch Curve	404
7.3	Fractal Dimensionality	406
7.4	Measuring Fractals	406
7.5	The Sierpinski Gasket	408
7.6	Other Ways to Define Fractals	408
7.7	Coding Fractals	409
7.8	The 2D Hilbert Curve	412
7.9	The 3D Hilbert Curve	416
	Exercises	418

8 Chaos 429

	Why Study Chaos?	429
8.1	Introduction	430
8.2	The Logistic Map	430

8.3 Stability and Period Doubling 436
8.4 The Lyapunov Exponent 438
8.5 The Feigenbaum Constants 439
8.6 Self-similarity Again 444
8.7 The Lorenz System 445
8.8 A Poincaré Section 448
Exercises 450

9 Computer Graphics Transformations 453
Why Study Computer Graphics Transformations? 453
9.1 Introduction 453
9.2 2D Transformations 454
9.3 Homogeneous Coordinates 455
9.4 Composition of 2D Transformations 459
9.5 Transforming a Knee 460
9.6 Multiple Transformations 465
9.7 Baravelle Spirals 466
9.8 Hilbert Curves Again 468
9.9 Matplotlib's Affine2D Classes 470
9.10 Affine2D Images 472
9.11 Affine2D Paths 473
9.12 Henderson's Functional Geometry 475
9.13 Hierarchy of Transformations 477
9.14 3D Transformations as Matrices 480
9.15 Composition of 3D Transformations 482
9.16 Recursive Cube Drawing 485
Exercises 488

10 Computational Geometry 493
Why Study Computational Geometry? 493
10.1 Introduction 493
10.2 Geometry Data Structures 494
10.3 The Dot and Cross Products 495
10.4 The ccw() function 498
10.5 Segment Intersection 499
10.6 Finding the Convex Hull 502
10.7 Finding the Closest Pair of Points 506
Exercises 510

11 Continued Fractions 513
Why Study Continued Fractions? 513
11.1 Introduction 513

11.2 Continued Fractions and Euclid's Algorithm 520
11.3 Continued Fractions for Square Roots 520
11.4 The Law of Best Approximation 522
Exercises 524

12 Discrete-Time Markov Chains 527
Why Study Discrete-Time Markov Chains (DTMCs)? 527
12.1 Introduction 527
12.2 *n*-Step Transitions 531
12.3 Simulation 533
12.4 Classification of States 536
12.5 Steady-State Behavior 537
12.6 The Two Urns 539
12.7 Absorption 540
12.8 Finding Absorbing Probabilities 542
12.9 Absorption Time and Probabilities 544
12.10 Implementing Absorption 545
12.11 Implementing Steady State Using Matrices 546
Exercises 548

13 Navigating the Earth 553
Why Study Spherical Trigonometry? 553
13.1 Introduction 553
13.2 Properties of the Sphere 554
13.3 Curves on a Sphere 555
13.4 Mapping a Sphere 555
13.5 Spherical Triangles 557
13.6 The Spherical Sine Law 559
13.7 Triangles made from Lunes 560
13.8 Are Great Circle Arcs the Shortest Distance? 562
13.9 Spherical Triangles Summarized 564
13.10 Calculating with Great Circles 564
13.11 The Mercator Projection 572
13.12 Calculating with Rhumb Lines 579
Exercises 585

14 Quaternions and Rotation 589
Why Study Quaternions? 589
14.1 Introduction 589
14.2 3D Rotation 590
14.3 Vectors Again 591
14.4 `Vec`: A vector class in Python 594

14.5 Rodrigues' Rotation Formula (RRF) 596
14.6 Quaternions 598
14.7 Rotations with Quaternions 602
14.8 A Quaternion class in Python 603
14.9 Spherical Linear Interpolation (Slerp) 606
14.10 Visualizing Lerp and Slerp 611
14.11 Navigating Again 613
14.12 Spherical Cubic Interpolation (Squad) 615
Exercises 619
Answers 620

15 Curve Fitting 623
Why Study Curve Fitting, Splines and Bézier curves? 623
15.1 Introduction 623
15.2 Newton Interpolation 624
15.3 Least Squares Curve Fitting 627
15.4 Cubic Spline Interpolation 628
15.5 The Bézier Curve 635
Exercises 649
Answers 650

16 Conic Sections 653
Why Study Conic Sections? 653
16.1 Introduction 653
16.2 A Geometric Definition 654
16.3 A Locus-Distance Definition 656
16.4 Classifying Conic Sections by Eccentricity 664
16.5 Polar Coordinates 666
16.6 Ellipses, Parabolas, and Hyperbolas Unified 666
16.7 Drawing Conic Sections using their Eccentricity 668
16.8 Dandelin Spheres 671
16.9 Quadratic Equations 675
16.10 Parameterized Conic Sections 680
16.11 Drawing Parameterized Equations 683
16.12 Reflections in Conic Sections 685
16.13 Confocal Conic Sections 693
Exercises 700
Answers 703

17 Solving a Cubic Equation 707
Why Study and Solve Cubic Equations? 707
17.1 Introduction 707

17.2 Depressing Cubic Equations 708
17.3 The Tartaglia-Cardano's Method 709
17.4 Viète's Method 713
17.5 Implementing Cardano's and Viète's Methods 715
17.6 Visually Finding Roots in a 2D Plot 718
17.7 Visually Finding Roots with a Modulus Surface 720
17.8 Visually Finding Roots with Enhanced Phase Maps 722

18 The Catenary 725
Why Study the Catenary? 725
18.1 The Catenary Chain 725
18.2 A Parabolic Chain 733
18.3 Visually Comparing the Catenary and Parabola 735
18.4 Determining the Catenary Parameter a 740
18.5 Using a Catenary to Calculate Logarithms 741
18.6 Rolling a Parabola 743
18.7 The Tractrix 746
18.8 Riding a Bike with Square Wheels 750
18.9 The Catenoid 757
Exercises 763
Answers 764

19 The Cycloid 767
Why Study the Cycloid? 767
19.1 The Cycloid 767
19.2 The Tautochrone 773
19.3 A Cycloid's Involute 777
19.4 The Brachistochrone 781
19.5 Epicycloids 786
19.6 Hypocycloids 790
19.7 Linking Hypocycloids and Epicycloids 793
19.8 Squaring the Circle 794
Exercises 795
Answers 796

20 Function Approximation 805
Why Study Function Approximation? 805
20.1 Dirty Functions as Motivation 806
20.2 Dirty Square Root 807
20.3 Dirty Log Base 10 812
20.4 Dirty Arctangent 820
20.5 Dirty Sine 824

20.6 The Maths of Function Approximation 828
20.7 The Taylor Series 829
20.8 The Lagrange Polynomial 842
20.9 Orthogonality 849
20.10 Chebyshev Approximation 854
20.11 Padé Approximation 861
Exercises 870
Answers 873

A A Crash Course in Probability 875
A.1 A Bold Gamble 879
A.2 Random Variables and Expectation 880
A.3 More on Random Walks 881
A.4 Five Practice Problems 882
A.5 More Information 885

B Solutions to Some Exercises 886
B.1 Sections 1.1 to 1.7 886
B.2 Section 2.1 889
B.3 Sections 2.2 to 2.4 893
B.4 Section 2.5 893
B.5 Section 2.5.4 893
B.6 Sections 2.6 to 2.12 894
B.7 Sections 2.13 to 2.14 895
B.8 Section 2.16 896
B.9 Chapters 1 to 2 896
B.10 Section 3.10 898
B.11 Section 3.11 898
B.12 Section 3.12 898
B.13 Section 3.15 899
B.14 Chapters 1 to 3 899
B.15 Sections 4.5 to 4.6 901
B.16 Section 4.13 901
B.17 Section 4.14 902
B.18 Chapters 3 to 4 902
B.19 Chapters 1 to 4 902
B.20 Section 5.4 903
B.21 Section 5.5.5 903
B.22 Section 5.6 904
B.23 Section 5.8 905
B.24 Section 6.2 905
B.25 Section 12 905

B.26 Chapters 1 to 7 909

Appendices 875

C Summary of Notation and Formulas 913

D A Summary of Python 915
D.1 Comments 915
D.2 Variables 915
D.3 Operators 916
D.4 Strings 916
D.5 I/O 917
D.6 Lists 917
D.7 Tuples 919
D.8 Dictionaries 919
D.9 Sets 919
D.10 If Statements 920
D.11 Loops 920
D.12 Functions 921
D.13 Exceptions 923
D.14 File I/O 923
D.15 Modules 923
D.16 Classes 924
D.17 Three Examples 924

E Numbers in Python 929
E.1 The int and float Types 929
E.2 The math Module 930
E.3 Numerical Errors 930
E.4 Representing Reals 931
E.5 Avoiding Float Problems 943
Exercises 947

F Matplotlib 950
F.1 The Big Three 952
F.2 Common Chart Variants 954
F.3 Customizing a Plot 957
F.4 Other Useful Plot Types 961
F.5 3D Plots 968
F.6 Animation 971
F.7 Widgets 973

G Turtle Graphics 975

G.1 Drawing Polygons 976
G.2 Responding to Mouse Clicks 977
G.3 Beautiful Recursion 978
G.4 Multiple Turtles 979
G.5 LOGO Flowers 980

H Graphviz 984
H.1 A Simple Undirected Graph 984
H.2 Using Nodes 986
H.3 More Layouts 990
H.4 More Attributes 991
H.5 Clusters: An Automaton 993
H.6 More Shapes 995

I Matrices 998
I.1 What is a Matrix? 998
I.2 Equality 999
I.3 Addition and Subtraction 999
I.4 Multiplication 1000
I.5 Transpose of a Matrix 1002
I.6 Special Matrices 1003
I.7 Determinant of a Square Matrix 1004
I.8 Inverse of a Square Matrix 1006
I.9 Solution of a Set of Linear Equations 1006
I.10 Gaussian Elimination 1007
I.11 Matrix Inversion Using Gaussian Elimination 1008

J Complex Numbers and `cmath` 1013
J.1 Multiplication in Polar Form 1015
J.2 De Moivre's Formula 1015
J.3 The *n*th Root of a Number 1016
J.4 Euler's Formula 1017
J.5 Multi-valued Exponentiation 1019
J.6 Complex Numbers in Python 1021

Bibliography 1029

Authors Bio 1035